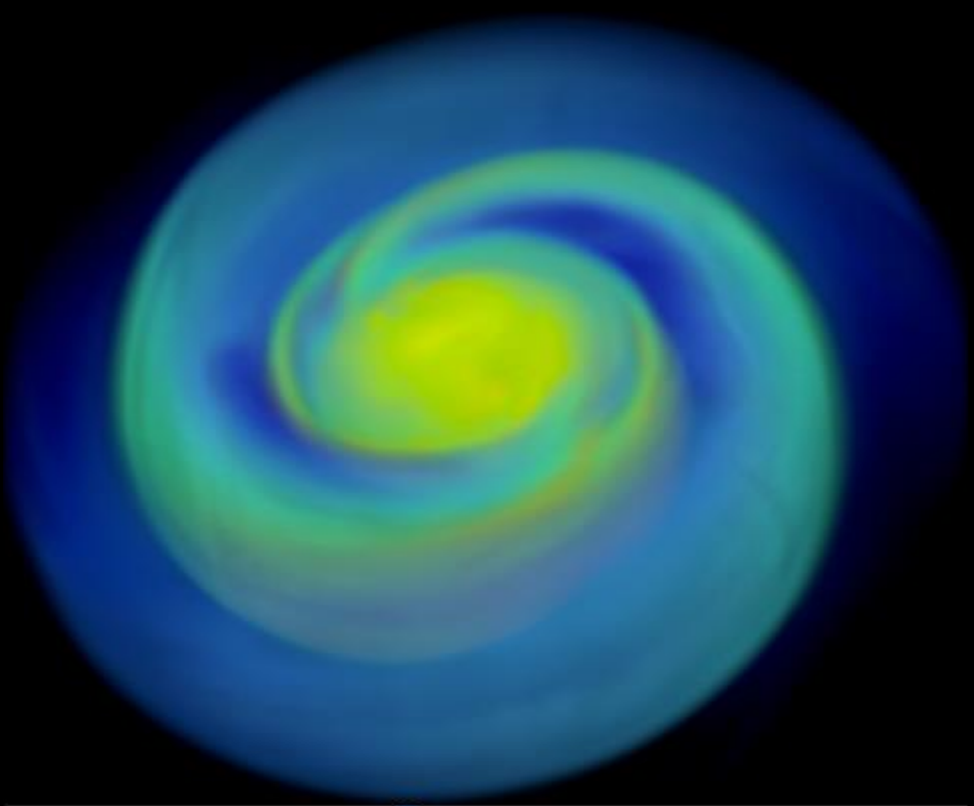
Two-moment transport in General Relativistic Magnetohydrodynamics simulations



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Boltzmann's equations

- Classical transport problem: evolve (6+1)D distribution function of neutrinos for each species:

$$p^\alpha \left[\frac{\partial f_\nu}{\partial x^\alpha} - \Gamma^i_{\alpha\gamma} p^\gamma \frac{\partial f_\nu}{\partial p^i}\right] = \nu \left[\frac{df_\nu}{d\tau}\right]_{\text{collisions}}$$

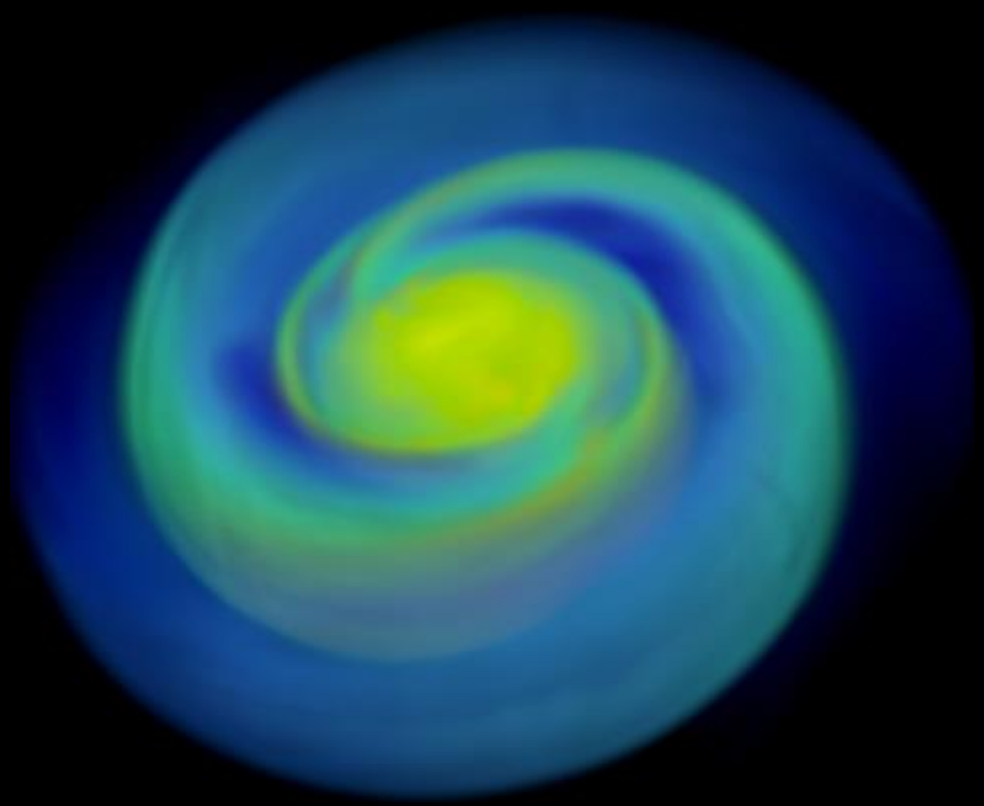
With

$$N = \int_V d^3x \frac{d^3p}{h^3} f_\nu(t, x^i, p_j)$$

- Collision terms include all reactions (calculated from non-linear, high-dimensional integrals over the neutrino and matter particles phase spaces), and is very stiff in dense regions
- Distribution is highly inhomogeneous

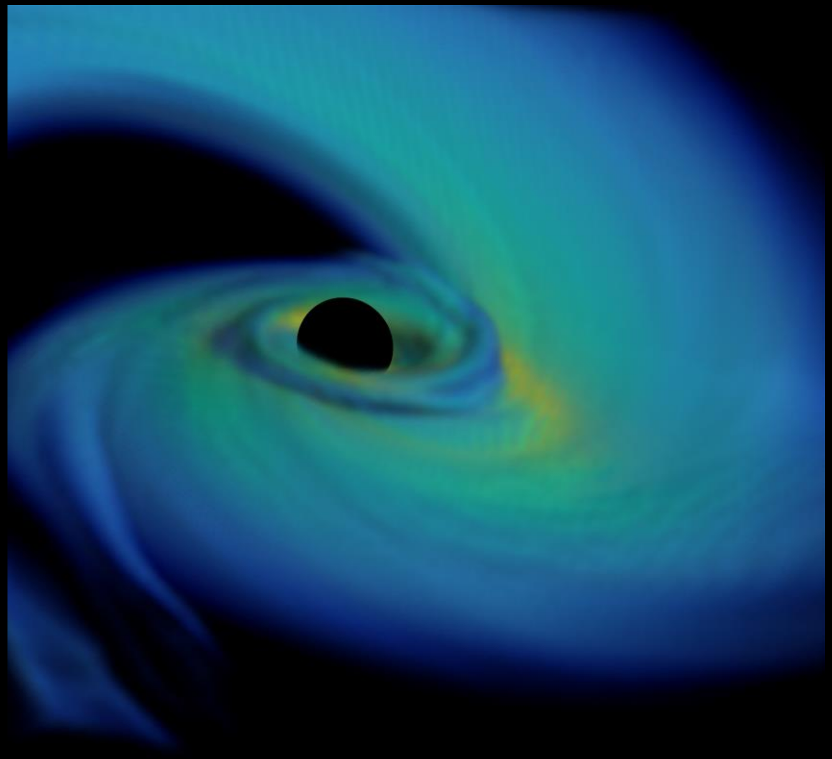
Transport in relativistic astrophysics: Two distinct regimes

With NS (incl. SNe):



Very high optical depth in NS

Without NS:



BH+semi-transparent disk

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Required interactions

Charged Current

$$n + e^+ \leftrightarrow p + \overline{\nu_e}$$
$$p + e^- \leftrightarrow n + \nu_e$$

+ Muons? Beta-decay?
Modified URCA?

Elastic Scattering

$$\nu + (p, n, N) \rightarrow \nu + (p, n, N)$$

+ inelastic scattering
on e⁻ ? Anisotropies?

Pair processes

$$e^+ + e^- \leftrightarrow \overline{\nu_x} + \nu_x$$
$$N + N \leftrightarrow N + N + \overline{\nu_x} + \nu_x$$
$$\gamma + \gamma \leftrightarrow \overline{\nu_x} + \nu_x$$

Oscillations

NMR, FFI,
Collisional…

Current calculations limited by dimensionality (inelastic scattering), incomplete angular distribution (pairs), resolution (oscillations), approximate reaction rates

Computing interaction rates

$$\int \Pi_{\{species\}} d^3p_s\,(f_1 f_2 \ldots)(1-f_3)(1-f_4)(\ldots)\delta(E_{in}-E_{out})\delta^3(\overline{p_{in}}-\overline{p_{out}})(\sigma_{int}\ldots)$$

Integral over 3S dimensions (S=number of participants in reaction)

Distribution function of incoming particles

Blocking factors for outgoing fermions

Energy/Momentum conservation

Cross section for particles of specific momenta

Add many-body corrections?

Result: Stiff coupling between neutrinos and fluid, with costly to compute collisional terms

$$p^\alpha \left[\frac{\partial f_\nu}{\partial x^\alpha} - \Gamma^i_{\alpha\gamma} p^\gamma \frac{\partial f_\nu}{\partial p^i}\right] = \nu \left[\frac{df_\nu}{d\tau}\right]_{\text{collisions}}$$

Particularly difficult if multiple particles have unknown distribution functions (i.e. are out of statistical equilibrium)!

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Moment formalism

- Integrate the distribution function of each species over momentum space:

$$E(t,x^i,\epsilon) = \epsilon^3 \int d\Omega f_\nu(t,x^i,\epsilon,\Omega)$$ Energy density

$$F^\alpha = \epsilon^3 \int d\Omega f_\nu(t,x^i,\epsilon,\Omega) l^\alpha$$ Energy flux density

$$P^{\alpha\beta} = \epsilon^3 \int d\Omega f_\nu(t,x^i,\epsilon,\Omega) l^\alpha l^\beta$$ Pressure

- Requires a choice of reference frame to decompose

$$p^\mu = \epsilon(\hat{t}^\mu + l^\mu)$$

- "Number weighted" moments can be obtained using ϵ^2 instead of ϵ^3

Moment Equations

- Integrate Boltzmann's equation for each species over momentum space and define

$$M^{\alpha\beta} = E\hat{t}^{\alpha}\hat{t}^{\beta} + F^{\alpha}\hat{t}^{\beta} + F^{\beta}\hat{t}^{\alpha} + P^{\alpha\beta}$$

to get the moment equations (2-moments, energy-weighted)

$$\nabla_{\beta}M^{\alpha\beta} - \frac{\partial}{\partial\nu}\left(\nu M^{\alpha\beta\gamma}\nabla_{\gamma}u_{\beta}\right) = S^{\alpha}$$

- The source terms are *(computed in the fluid frame!):*

$$\left[\frac{df_{\nu}}{d\tau}\right]_{\text{collisions}} = S(t, x^i, \nu, \Omega, f_{\nu})$$

$$S^{\alpha}(t, x^i, \nu) = \nu^3 \int d\Omega S(t, x^i, \nu, \Omega, f_{\nu})(u^{\alpha} + l^{\alpha})$$

- Similar equations can be derived for higher-order moments and/or number weighted moments
- Equations are ***exact*** but ***underconstrained***
- Calculation of S may require knowledge of the full distribution functions and couples all species / energies

Coupling to fluid

- Neutrino stress-energy tensor:

$$T_\nu^{\alpha\beta} = \int d\nu M^{\alpha\beta}(\nu)$$

- Energy-momentum conservation:

$$\nabla_\beta\left(T_\nu^{\alpha\beta} + T_{fluid}^{\alpha\beta}\right) = 0 \quad \Rightarrow \quad \nabla_\beta\left(T_{fluid}^{\alpha\beta}\right) = -\int d\nu S^\alpha(\nu)$$

- Lepton number conservation:

$$\partial_t(\rho_* Y_e) + \partial_i\left(\rho_* Y_e v_T^i\right) = m_b \alpha \left(\tilde{S}_{N,(\bar{\nu}_e)} - \tilde{S}_{N,(\nu_e)}\right) \qquad S_N = \int_0^\infty d\nu \nu^2 \int d\Omega S(t, x^i, \nu, \Omega, f_\nu)$$

… and again, S may not be possible to compute from moments only

Moment Equations : Gray Scheme

- Integrate moment equation over neutrino energy

$$\Rightarrow \nabla_\beta \left(T_s^{\alpha\beta}\right) = \int d\nu S_s^\alpha(\nu)$$

- In 3+1 form, leads to "fluid-like" flux-conservative equations:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\partial_t \tilde{E} + \partial_i \left(\alpha \tilde{F}^i - \beta^i \tilde{E}\right) = \alpha \tilde{P}^{ij} K_{ij} - \tilde{F}^j \partial_j \alpha - \alpha \tilde{S}^\mu n_\mu$$

$$\partial_t \tilde{F}_j + \partial_i \left(\alpha \tilde{P}_j^i - \beta^i \tilde{F}_j\right) = -\tilde{E}\partial_j \alpha + \tilde{F}_k \partial_j \beta^k + \frac{\alpha}{2}\tilde{P}^{ik}\partial_j \gamma_{ik} + \alpha \tilde{S}^\mu \gamma_{j\mu}$$

- Only P^{ij} moment is unknown
- Coupling to fluid stress-energy tensor remains simple
- … but sources S^μ, S_N are heavily dependent on assumed neutrino spectrum!

Improved Gray Scheme

- Potential improvement: evolve number density

$$\partial_t \tilde{N} + \partial_i \left(\alpha \tilde{F}^i_N - \beta^i \tilde{N} \right) = \alpha \tilde{S}_N$$

- … and make a "reasonable" approximation for the number flux
- Provides an estimate of the neutrino average energy at least

Sketch of numerical algorithm

$$\nabla_\beta M^{\alpha\beta} - \frac{\partial}{\partial\nu}\left(\nu M^{\alpha\beta\gamma}\nabla_\gamma u_\beta\right) = S^\alpha$$

- Assume we evolve E, F^i either in gray or energy-dependent scheme
- Compute closure for higher-order moments
- Compute spatial fluxes (and energy fluxes if used)
- Time stepping requires treating source terms implicitly and:
  - S depends on *fluid-frame* moments
  - S requires energy closure for gray scheme, higher-order moments for some reactions
  - S couples all species of neutrinos at all energy group, as well as the fluid!
  - Common simplification: use gray scheme, guess at final fluid properties, approximate S as function of low-order moments

Pressure closure

- Most common choice for two-moment schemes: **Minerbo closure**
- Optically thick limit: use known pressure tensor $P_{ij}^{thick} = \frac{1}{3} E\ \delta_{ij}$
- Optically thin limit: assume all neutrinos move in direction of flux F^i, which gives $P_{ij}^{thin} = \frac{F_i F_j}{E}$ with the implicit assumption that $E = \sqrt{(F^i F_i)}$
- In between, interpolation $\mathrm{P_{ij}} = \xi P_{ij}^{thick} + (1-\xi) P_{ij}^{thin}$ with $\xi = f(\frac{J}{\sqrt{H^\alpha H_\alpha}})$
- Only correct for problems with spherical or planar symmetry, but can be reliable in optically thick regions
- ***Other choices are possible:*** MC closure, short characteristics, Eddington,… each with their own pros and cons…

Pressure closure -- Minerbo

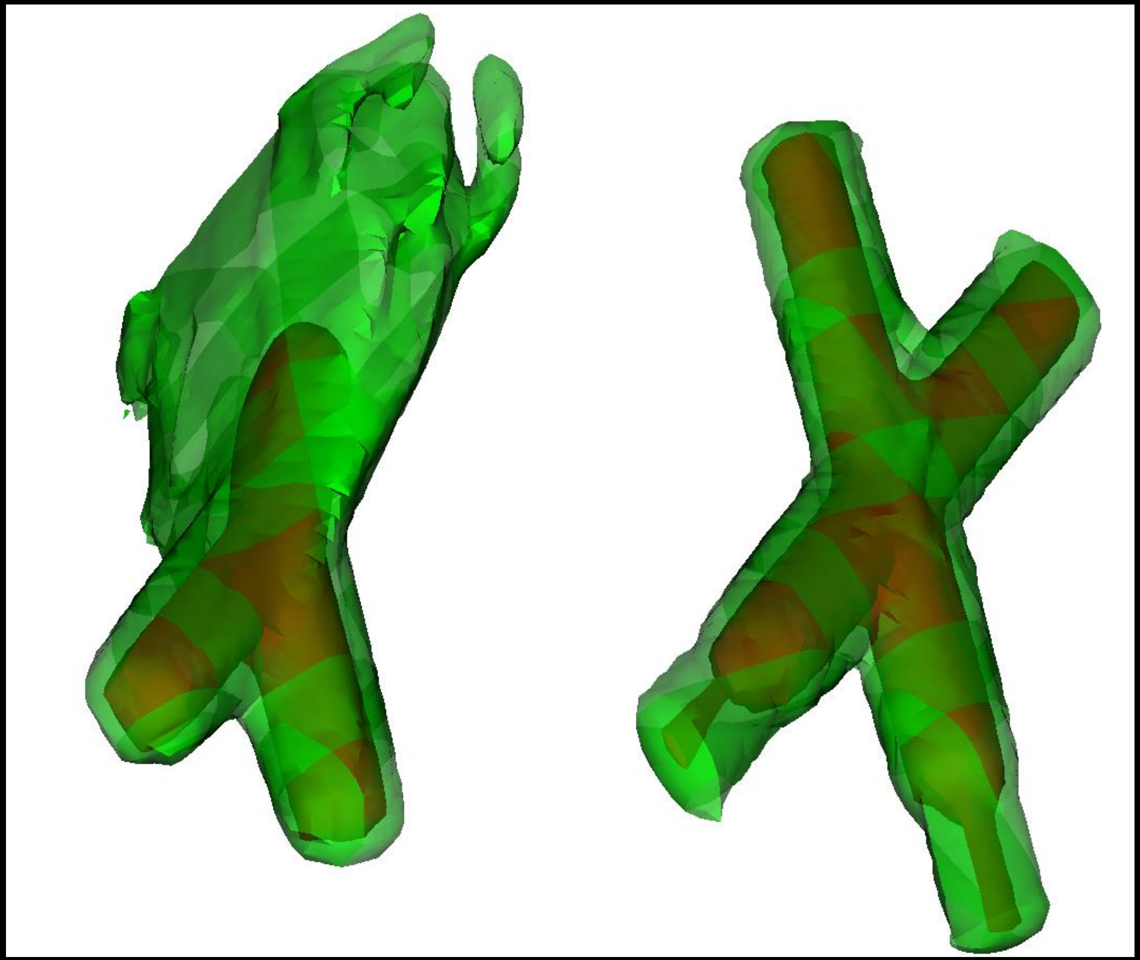


Image: Foucart *et al* (2018)

Interaction rates: Kirchoff's law

- Most simulations attempt to explicitly choose interaction rates such that the equilibrium distribution function giving $S = 0$ matches theoretical expectations (Fermi-Dirac distribution)

$$f_\nu^{eq} = \frac{1}{1 + \exp(\frac{\nu - \mu}{k_B T})}$$

- We can e.g. compute emission rates and calculate absorption rates to get the right equilibrium (or vice-versa)
- Do this for both the energy density and the number density

$$E^{eq} = \frac{\int d^3p\, \nu f_\nu^{eq}}{h^3} \;;\; N^{eq} = \frac{\int d^3p\, f_\nu^{eq}}{h^3}$$

Approximate sources

- Common choice for the form of the sources:
$$S^\alpha = \eta u^\alpha - \kappa_a J\, u^\alpha - \left(\kappa_a + \kappa_s^{el}\right) H^\alpha$$
including emissivity, absorption, and elastic scattering only.
- Can add inelastic scattering for energy-dependent moments and
$$S_N = \eta_N - \kappa_N N$$
for gray schemes.
- **This is an approximation!** E.g. pair processes can't be well described in this way…
- Depends on fluid frame moments, not evolved moments!

Energy closure

- For gray schemes, knowing $\eta(\nu)$ is not sufficient… we need to integrate over the neutrino spectrum
- In optically thick regions, use equilibrium spectrum at fluid temperature
- In optically thin regions, we need a closure in energy space!
  - Most reactions approximately scale as ν^2 (but see Andresen ’24 to do better)
  - Assume equilibrium at fluid temperature everywhere
    - Pretty bad, as neutrinos have effective temperature of (5-10)MeV, fluid O(1MeV)
  - Assume equilibrium spectrum consistent with evolved (E,N)
    - Better, ignores effects of diffusion
  - Ad-hoc corrections
    - Problem specific, uncontrolled errors
  - Advanced closure (MC, short-characteristics,…)
    - More costly…

Energy closure

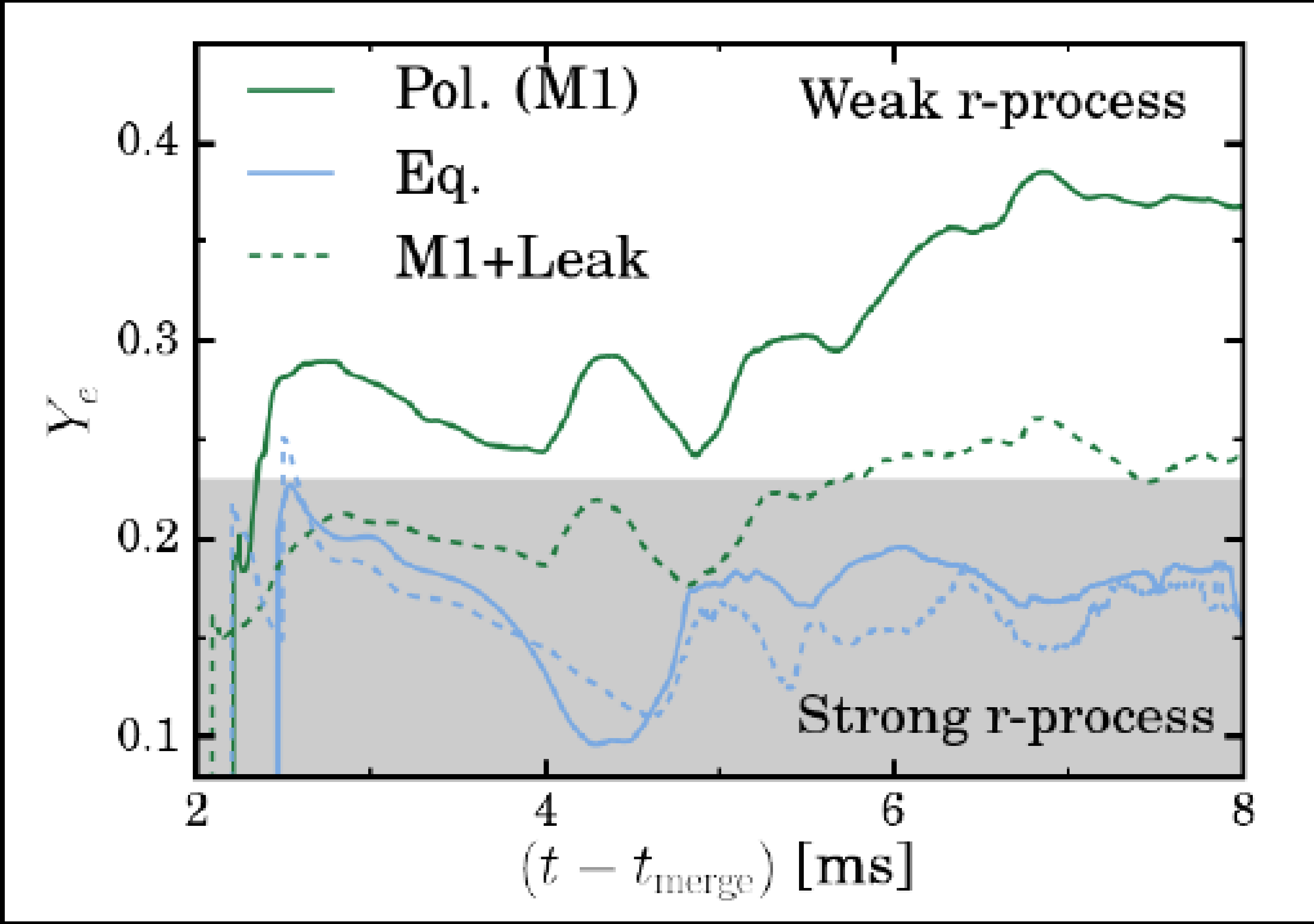


Image: Foucart *et al* (2017)

Implicit time stepping

- In theory, time step involves an implicit solve over all evolved moments and 5 fluid variables (energy, 3-momentum, lepton number)
  - Two-moments: $4\, N_{species} N_{groups} + 5$ variables. Very expensive.
- Common strategy: split time step
  - Solve $\partial_t U_{fluid} + \partial_i F^i_{fluid} = S_{fluid}$ to evolve fluid variables
  - Guess fluid variables U^*after coupling to neutrinos (multiple methods for this)
  - Evolve neutrino moments, *at fixed fluid variables* U^*
    - *If inelastic scattering / pairs can be treated explicitly* implicit solve done by block of 4 variables (E, F^i) *for each species/group*
    - *Use implicit-explicit scheme,*
      $\partial_t U_\nu = S_{explicit} + \partial_i F^i_\nu + S_{implicit}$
      $\rightarrow U^{n+1} = U^n + dt\, [S_{explicit}(U^n) + \partial_i F^i_\nu(U^n) + S_{implicit}(U^{n+1})]$
  - Update fluid variables using interaction rates from moment evolution

Numerical fluxes

- So far, we have ignored the $\partial_i F_\nu^i$ term in the moment equation
  - Standard algorithm: use shock capturing methods, identical to hydrodynamics: MC/PPM/WENO5 with LLF or HLL Riemann solver
  - Specifically,

    $$\partial_i F^i(\bar{x}) = \frac{F^{*i}\left(\bar{x}+\frac{\Delta x}{2}\right)-F^{*i}\left(\bar{x}-\frac{\Delta x}{2}\right)}{\Delta x}$$

    with F^* a numerical flux, e.g. the LLF flux

    $$F^* = \frac{F^+ + F^-}{2} - c_{max}\frac{U^+ - U^-}{2}$$

    … but this introduces diffusion due to the dissipation term
- Two solutions
  - Explicitly set the flux to its expected value in the diffusion limit (Audit)
  - Combine shock-capturing fluxes with less dissipative methods (Radice)

Diffusion regime

- Accurate solve of the implicit problem and flux corrections in high-opacity regimes are important to the quality of solutions in the diffusive regime.

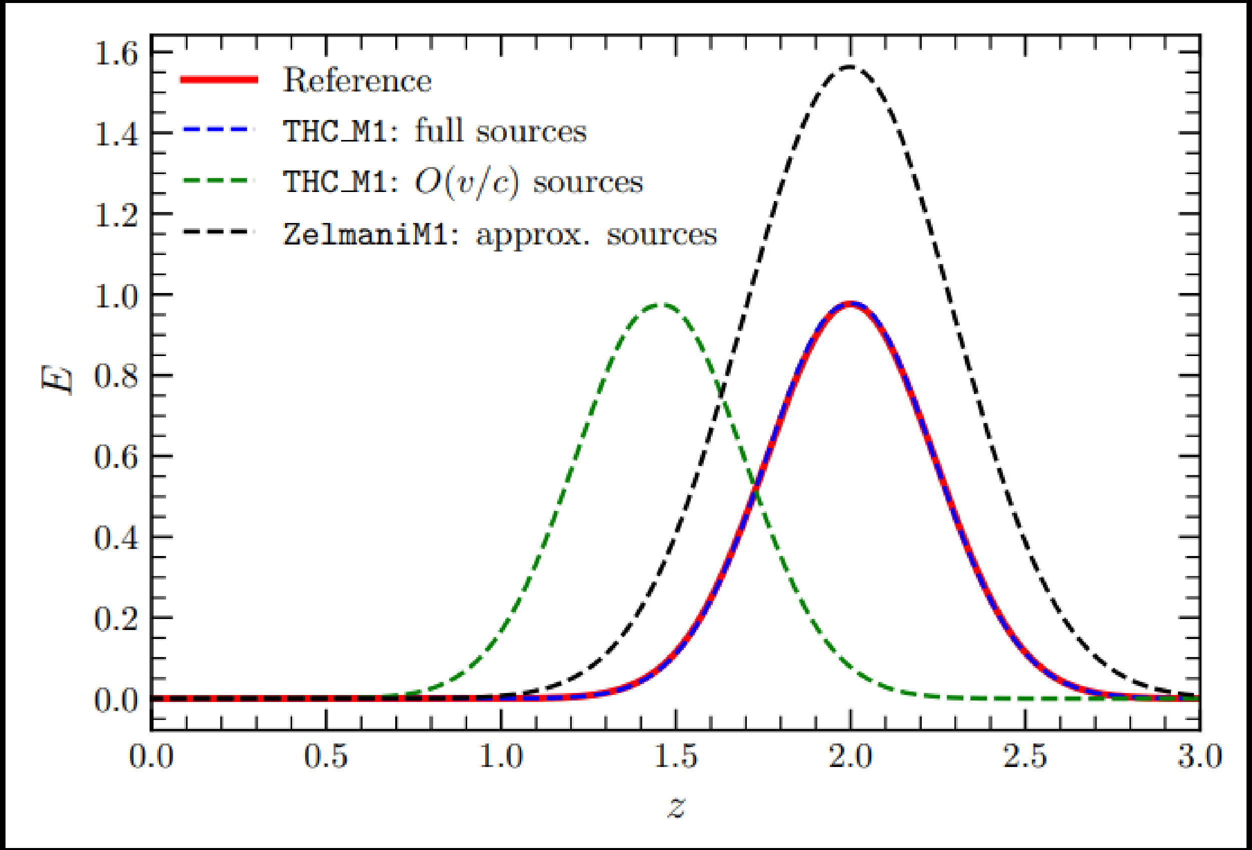
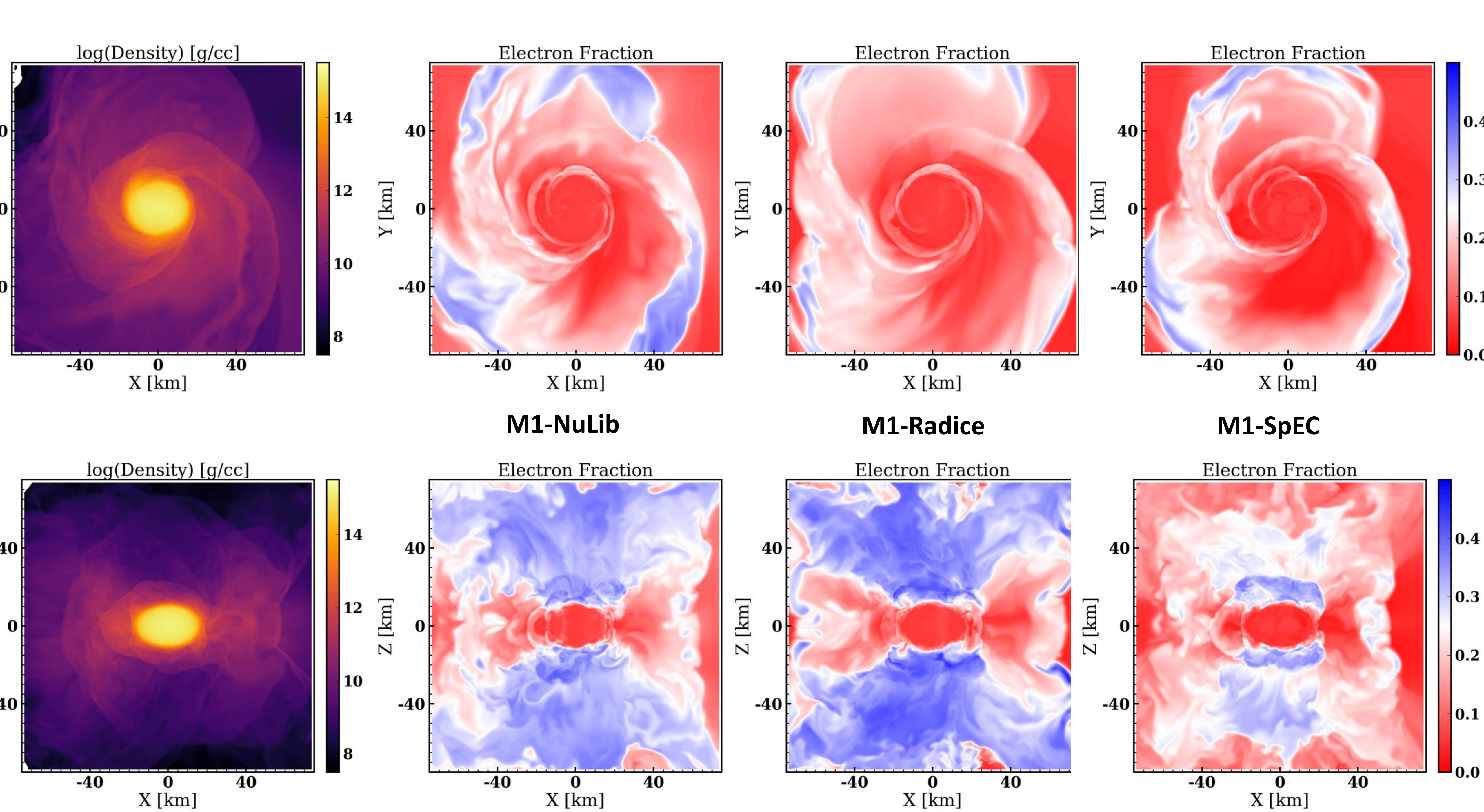


Image: Radice *et al* (2022)


log(Density) [g/cc]
Electron Fraction
Electron Fraction
Electron Fraction
M1-NuLib
M1-Radice
M1-SpEC
log(Density) [g/cc]
Electron Fraction
Electron Fraction
Electron Fraction
X [km]
Y [km]
Z [km]

"Missing" reactions

- Pair annihilation combine distribution function of two neutrino species, and are sensitive to the pressure closure
  - Dominant emission source for heavy-lepton neutrinos
  - Dominant source of energy deposition in polar regions
- Inelastic scattering on electrons may change neutrino spectrum
  - Not attempted in gray scheme
  - Couples neutrinos of different energies in spectral moment schemes
- Oscillations likely active close to the merger remnant (FFI, NMR,…)
  - FFI grows on cm scales!
  - Requires evolution of a 3x3 density matrix instead of the distribution function
  - Only included very approximately or studied in post-processing so far